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A reduced BPS index of E-strings

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ABSTRACT: We study the BPS spectrum of E-strings in a situation where the global E_8 symmetry is broken down to $D_4 \oplus D_4$ by a certain twist. We find that the refined BPS index in this setup serves as a reduced BPS index of E-strings, which gives a novel trigonometric generalization of the Nekrasov partition function for four-dimensional $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with $N_f = 4$ massless flavors. We determine the perturbative part of this index and also first few unrefined instanton corrections. By using these results together with the modular anomaly equation, the genus expansion of the free energy can be computed efficiently up to very high order. We also determine the unrefined elliptic genus of four E-strings under the twist.

KEYWORDS: Supersymmetric gauge theory, Field Theories in Higher Dimensions, Topological Strings

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1 Introduction

The E-string theory is one of the simplest interacting supersymmetric field theories in six dimensions [1–5]. The theory arises from an M5-brane probing the end-of-the-world 9-plane in the heterotic M-theory. The fundamental objects of the theory are called E-strings and are realized by M2-branes stretched between the M5-brane and the 9-plane. While the theory itself has begun to attract revived interests [6, 7], the BPS spectrum of E-strings has been drawing continuous attention. One reason for this is that the BPS spectrum of toroidally compactified E-string theory encompasses that of various gauge theories with rank-one gauge groups in five and four dimensions [4]. Another reason is that the BPS index of E-strings is essentially equivalent to the topological string partition function for the local $\frac{1}{2}$ K3 Calabi-Yau threefold [3, 5], which is a key example to study topological strings on non-toric Calabi-Yau threefolds.

In this paper we focus on a particular twist of the E-string theory. We consider a situation where the global E_8 symmetry is broken down to $D_4 \oplus D_4$. In other words, we consider a Seiberg-Witten system [8, 9] which has one-dimensional Coulomb branch moduli space with two D_4 -type singularities. This is very reminiscent of the moduli space of four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with $N_f = 4$ massless flavors [9]. In fact, concerning the moduli space the only difference is that in our setup two D_4 singularities are placed at a finite distance from each other, whereas in the $SU(2)$ $N_f = 4$ theory one of the D_4 singularities is at infinity. A peculiar feature common to these Seiberg-Witten systems is that the complex structure of the Seiberg-Witten curve, which represents the gauge coupling, is constant all over the moduli space. This leads to substantial simplification of the BPS spectra. Nevertheless, the resulting spectra are not boring. In fact, it is well known that the Nekrasov partition function for the $SU(2)$ $N_f = 4$ theory [10, 11] is nontrivial even in the massless limit [12]. The main objective of this paper is to clarify the counterpart of this function on the E-string theory side.

More specifically, we study the refined BPS index of E-strings evaluated under the $D_4 \oplus D_4$ twist. We find that this serves as a reduced BPS index of E-strings. A characteristic feature of the reduced index is that it inherits the modular properties from the original BPS index of E-strings. In particular, the reduced index satisfies the same modular anomaly equation as that of the original BPS index [13, 14]. There is another function which also possesses such modular characteristics: the refined BPS index of E-strings with untwisted E_8 symmetry. Interestingly, our reduced BPS index is even simpler than this function and thus easier to deal with. Hence the reduced index would serve as a convenient tool for understanding the whole structure of the BPS spectrum of E-strings.

We study in detail this reduced BPS index by performing three kinds of expansions. One is the genus expansion of topological strings, another is the q -expansion, which is closely related to the spacetime instanton expansion, and the other is the winding number expansion, which generates elliptic genera of multiple E-strings. Making use of the modular anomaly equation and comparing these expansions with one another, we are able to clarify the structure of the expansion as well as to determine some expansion coefficients. Among others, an intriguing outcome of this study is that the reduced BPS index gives a novel trigonometric generalization of the Nekrasov partition function for 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ theory with $N_f = 4$ massless flavors.

The paper is organized as follows. In section 2, we present the definition of the reduced BPS index and summarize its general properties. In section 3, we consider the genus expansion of the free energy and study topological string amplitudes for the local $\frac{1}{2}K3$ Calabi-Yau threefold evaluated under the $D_4 \oplus D_4$ twist. We clarify the general structure of these amplitudes and discuss how to determine them. In section 4, we consider the instanton expansion of the free energy. Our conjectures about the perturbative part and first few unrefined instanton corrections are presented. We discuss their structures, in particular in relation to Nekrasov partition functions for ordinary gauge theories. In section 5, we consider the winding number expansion and study elliptic genera of multiple E-strings. The elliptic genus of two E-strings is presented in a very simple form. We also discuss the structure of general unrefined elliptic genera and determine the unrefined elliptic genus

of four E-strings. In section 6, we elucidate how the reduced BPS index reproduces the Nekrasov partition function for 4d $\mathcal{N} = 2$ SU(2) gauge theory with $N_f = 4$ massless flavors. In section 7 we discuss possible extensions of this work. In appendix A, we summarize our conventions of special functions and present some useful formulas.

2 Definition and general properties

2.1 Brief review of refined BPS index of E-strings

In this subsection we briefly review the refined BPS index of E-strings. Consider the E-string theory on $\mathbb{R}^5 \times S^1$. One can regard this theory effectively as a 5d $\mathcal{N} = 1$ supersymmetric field theory, where BPS E-strings wound around S^1 are viewed as 5d BPS particles. The refined BPS index of E-strings is defined as the 5d BPS index [10, 15] for this toroidally compactified E-string theory:

$$Z^{\text{gen}}(\phi, \tau, \mathbf{m}, \epsilon_1, \epsilon_2) := \text{Tr}(-1)^{2J_L+2J_R} y_L^{J_L} y_R^{J_R+J_I} p^n q^k e^{i\mathbf{\Lambda} \cdot \mathbf{m}}, \quad (2.1)$$

where

$$y_L := e^{i(\epsilon_1 - \epsilon_2)}, \quad y_R := e^{i(\epsilon_1 + \epsilon_2)}, \quad p := e^{-\phi}, \quad q := e^{2\pi i \tau}. \quad (2.2)$$

Here J_L, J_R, J_I and $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_8)$ are spins (or weights of the associated Lie algebras) of the little group $\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$, the R-symmetry group $\text{SU}(2)_I$ and the global symmetry group E_8 respectively. Nonnegative integers n, k are respectively the winding number and the momentum along S^1 . As we see the above index is a function in twelve variables: ϕ is the tension of the E-strings, τ is proportional to the inverse of the radius of S^1 , $\mathbf{m} = (m_1, \dots, m_8)$ and ϵ_1, ϵ_2 are respectively the Wilson line parameters for the global symmetries E_8 and $\text{SO}(4)$.

The BPS index Z^{gen} can be viewed as the generating function for the sequence of elliptic genera of multiple E-strings. It is expanded as

$$Z^{\text{gen}} = 1 + \sum_{n=1}^{\infty} p^n Z_n^{\text{gen}}, \quad (2.3)$$

where Z_n^{gen} is (the holomorphic limit of) the elliptic genus of n E-strings. The elliptic genus of single E-strings is known as [14]

$$Z_1^{\text{gen}} = -q^{1/2} \frac{A_1(\tau, \mathbf{m})}{\eta^6 \vartheta_1(\epsilon_1) \vartheta_1(\epsilon_2)}. \quad (2.4)$$

Here $A_1(\tau, \mathbf{m}) := \frac{1}{2} \sum_{k=1}^4 \prod_{j=1}^8 \vartheta_k(m_j)$ is the theta function of the E_8 root lattice. The elliptic genus of two E-strings Z_2^{gen} was also calculated recently [16]. In general, Z_n^{gen} is some $\text{Weyl}(E_8)$ -invariant Jacobi form of index n with respect to \mathbf{m} and can be expressed in terms of nine generators [17]: $A_k(\tau, \mathbf{m})$ with $k = 1, 2, 3, 4, 5$ and $B_k(\tau, \mathbf{m})$ with $k = 2, 3, 4, 6$, where k represents the index of the Jacobi form, or, the level of the associated E_8 current algebra.

The BPS index can also be viewed as the instanton part of the refined topological string partition function for the local $\frac{1}{2}\text{K3}$ Calabi-Yau threefold:

$$Z^{\text{gen}} = \exp F^{\text{gen}} \quad (2.5)$$

with

$$F^{\text{gen}} = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1 \epsilon_2)^{g-1} F^{\text{gen}(n,g)}. \quad (2.6)$$

Topological string amplitudes for the local $\frac{1}{2}\text{K3}$ at low genus have been studied extensively. See e.g. [14, 17] for the latest results of $F^{\text{gen}(n,g)}$ with general \mathbf{m} . In general, a topological string amplitude at low genus has its classical part, which is some polynomial at most cubic in the Kähler moduli parameters. In this paper we adopt (2.1) as the primary definition of Z^{gen} and let $F^{\text{gen}(n,g)}$ be made up of purely world-sheet instanton contributions.

2.2 Reduced BPS index and $D_4 \oplus D_4$ twist

In this paper we consider a BPS index of E-strings defined as

$$Z(\phi, \tau, \epsilon_1, \epsilon_2) := \text{Tr}(-1)^{2J_L + 2J_R + \Lambda_3 + \Lambda_4 - \Lambda_5 - \Lambda_6} y_L^{J_L} y_R^{J_R + J_I} p^n q^{k + (-\Lambda_5 - \Lambda_6 + \Lambda_7 + \Lambda_8)/2}. \quad (2.7)$$

This is equivalent to the refined BPS index (2.1) evaluated at the following special values of the E_8 Wilson line parameters

$$m_1 = m_2 = 0, \quad m_3 = m_4 = \pi, \quad m_5 = m_6 = -\pi - \pi\tau, \quad m_7 = m_8 = \pi\tau. \quad (2.8)$$

This setting of Wilson line parameters corresponds to breaking the global symmetry E_8 down to $D_4 \oplus D_4$. This is manifestly seen in the Seiberg-Witten description: the Seiberg-Witten curve for the E-string theory [18] with the above parameter values reduces to the following very simple form [19]

$$y^2 = 4x^3 - \frac{E_4}{12} (u^2 - u_0^2)^2 x - \frac{E_6}{216} (u^2 - u_0^2)^3, \quad (2.9)$$

where

$$u_0 := -\frac{2}{q^{1/2}\eta^{12}}. \quad (2.10)$$

The curve describes an elliptic fibration over the moduli space parametrized by u , with D_4 type singularities at $u = \pm u_0$. It is easy to see that the complex structure of the above curve is τ and does not depend on u .

As we mentioned, the refined BPS index of E-strings is expressed in terms of nine $\text{Weyl}(E_8)$ -invariant Jacobi forms A_k, B_k . Under the twist (2.8), they become

$$\begin{aligned} A_1 = A_3 = A_5 = B_3 &= 0, \\ A_2 &= \frac{E_4}{9q}, & A_4 &= \frac{E_4}{q^2}, \\ B_2 &= -\frac{E_6}{15q}, & B_4 &= -\frac{E_6}{15q^2}, & B_6 &= \frac{E_6}{9q^3}. \end{aligned} \quad (2.11)$$

There are several known results about Z^{gen} expressed in terms of these Jacobi forms [14, 16, 17]. These results can be immediately translated for our reduced BPS index Z by simply substituting (2.11). We will do this in the following sections. Note that all Jacobi forms of odd index vanish under the twist. Nonvanishing Jacobi forms also take very simple forms. This leads to substantial simplification of the refined BPS index.

2.3 Expansions

In the following sections we will study the index Z by performing three kinds of expansions. First, as in the case of Z^{gen} , the free energy

$$F = \ln Z \quad (2.12)$$

admits the genus expansion

$$F = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1 \epsilon_2)^{g-1} F^{(n,g)}(\phi, \tau). \quad (2.13)$$

Here $F^{(n,g)}$ are interpreted as topological string amplitudes evaluated under the twist (2.8). We will study them in section 3.

It is also interesting to consider the instanton expansion defined as follows

$$F = F^{\text{pert}}(\phi, \epsilon_1, \epsilon_2) + \sum_{k=1}^{\infty} q^k F_k^{q\text{-inst}}(\phi, \epsilon_1, \epsilon_2). \quad (2.14)$$

This expansion is closely related to the instanton expansion of conformal gauge theories. We will study this expansion in section 4.

The other expansion is the winding number expansion

$$Z = 1 + \sum_{n=1}^{\infty} p^{2n} Z_{2n}(\tau, \epsilon_1, \epsilon_2). \quad (2.15)$$

Z_{2n} is, as before, the elliptic genus of $2n$ E-strings evaluated under the $D_4 \oplus D_4$ twist. Z_k with odd k are absent because $\text{Weyl}(E_8)$ -invariant Jacobi forms of odd index vanish under the $D_4 \oplus D_4$ twist, as we saw in (2.11). We will study these elliptic genera in section 5.

2.4 Modular anomaly equation

The BPS index Z^{gen} for E-strings is known to satisfy the modular anomaly equation [13, 14]. As one can see from (2.11), the $D_4 \oplus D_4$ twist does not modify the structure of modular anomalies, i.e. the way how E_2 enters in the index. Thus, our reduced BPS index Z satisfies the same modular anomaly equation

$$\partial_{E_2} Z = -\frac{1}{24} [\epsilon_1 \epsilon_2 \partial_{\phi} (\partial_{\phi} - 1) + (\epsilon_1 + \epsilon_2)^2 \partial_{\phi}] Z. \quad (2.16)$$

In terms of $F^{(n,g)}$, it is expressed as

$$\begin{aligned} \partial_{E_2} F^{(n,g)} &= \frac{1}{24} \sum_{n_1=0}^n \sum_{g_1=0}^g \partial_{\phi} F^{(n_1, g_1)} \partial_{\phi} F^{(n-n_1, g-g_1)} \\ &\quad + \frac{1}{24} \partial_{\phi} (\partial_{\phi} - 1) F^{(n, g-1)} - \frac{1}{24} \partial_{\phi} F^{(n-1, g)}, \end{aligned} \quad (2.17)$$

where $F^{(n,g)} = 0$ if $n < 0$ or $g < 0$. We will use this form of the equation in section 3. In terms of the elliptic genera Z_{2n} the modular anomaly equation is simply

$$\partial_{E_2} Z_{2n} = \frac{1}{24} [-2n(2n+1)\epsilon_1 \epsilon_2 + 2n(\epsilon_1 + \epsilon_2)^2] Z_{2n}, \quad (2.18)$$

which we will use in section 5.

3 Genus expansion

In this section we study the genus expansion (2.13). The expansion coefficients $F^{(n,g)}$ represent topological string amplitudes for the local $\frac{1}{2}\text{K3}$ evaluated under the $D_4 \oplus D_4$ twist. Explicit forms of general amplitudes $F^{\text{gen}(n,g)}$ at low genus are known [17]. There $F^{\text{gen}(n,g)}$ are expressed in terms of period integrals over the Seiberg-Witten curve, which are related to the coordinate of the moduli space by the mirror map. Thus, we first summarize in subsection 3.1 how the mirror map and the period integrals are reduced under the $D_4 \oplus D_4$ twist.

To determine higher genus amplitudes, the modular anomaly equation (2.17) works as a powerful tool. In subsection 3.2, we derive the general structure of $F^{(n,g)}$ and explain how to determine the modular ambiguities of $F^{(n,g)}$ at high genus.

3.1 Mirror map

The purpose of this subsection is to present how the mirror map and the period integrals studied in [17] are reduced under the $D_4 \oplus D_4$ twist. We intend this subsection to be a short summary of the results. The reader is referred for definitions of the symbols used here and in (3.9) to [17]. Note also that the variable ϕ used in [17] should not be confused with ϕ in this paper. They are related with each other by

$$\phi_{\text{there}} = -\phi_{\text{here}} + \ln(-q^{1/2}\eta^{12}). \quad (3.1)$$

As we mentioned, the gauge coupling $\tilde{\tau}(u)$ of the low energy effective theory is constant all over the moduli space

$$\tilde{\tau} = \tau. \quad (3.2)$$

Due to this, the mirror map takes a very simple form. One obtains

$$\omega = (u^2 - u_0^2)^{-1/2}, \quad (3.3)$$

$$p = e^{-\phi} = \frac{u_0}{2} e^{\phi_{\text{there}}} = \frac{u_0}{u + \sqrt{u^2 - u_0^2}}. \quad (3.4)$$

These relations can also be written as

$$u = u_0 \cosh \phi, \quad (3.5)$$

$$\omega = \frac{1}{u_0 \sinh \phi}. \quad (3.6)$$

It then follows that

$$\tilde{\Delta} = \Delta, \quad \tilde{E}_{2n} = E_{2n}, \quad (3.7)$$

$$\ln \omega = \phi_{\text{there}} - \ln(1 - p^2). \quad (3.8)$$

3.2 Topological string amplitudes

In [17], explicit forms of $F^{\text{gen}(n,g)}$ with $n = 0$, $0 \leq g \leq 3$ were calculated. It is straightforward to generalize the calculation to the refined case. For our present purposes it is sufficient to compute the explicit form of $F^{\text{gen}(1,0)}$ in addition. The result is

$$F^{\text{gen}(1,0)} = \frac{1}{2} \ln \omega - \frac{1}{24} \ln \tilde{\Delta} + \frac{1}{24} \ln \Delta - \frac{1}{2} \phi_{\text{there}}. \quad (3.9)$$

Using the relations (3.7), (3.8), one can translate these results of $F^{\text{gen}(n,g)}$ into

$$\begin{aligned} F^{(0,0)} &= 0, \\ F^{(0,1)} &= F^{(1,0)} = -\frac{1}{2} \ln(1 - e^{-2\phi}), \\ F^{(0,2)} &= \frac{1}{32} \frac{E_2}{\sinh^2 \phi}, \\ F^{(0,3)} &= \frac{1}{768} \frac{3E_2^2 + E_4}{\sinh^2 \phi} + \frac{1}{384} \frac{2E_2^2 + E_4}{\sinh^4 \phi}. \end{aligned} \quad (3.10)$$

These results give us sufficient information to derive the general structure of $F^{(n,g)}$. Given the modular anomaly equation (2.17) and the explicit forms of $F^{(0,0)}$, $F^{(0,1)}$, $F^{(1,0)}$, it is not difficult to show that $F^{(n,g)}$ with $n + g \geq 2$ has to take the following form

$$F^{(n,g)}(\phi, \tau) = \sum_{h=1}^{n+g-1} \frac{F^{(n,g,h)}(\tau)}{\sinh^{2h} \phi}, \quad (3.11)$$

where

$$F^{(n,g,h)}(\tau) = [\text{quasi modular form of weight } 2n + 2g - 2]. \quad (3.12)$$

As an illustration, we present the explicit forms of $F^{(n,g)}$ with $n + g = 2, 3$:

$$\begin{aligned} F^{(2,0)} &= \frac{1}{96} \frac{E_2}{\sinh^2 \phi}, & F^{(1,1)} &= \frac{1}{24} \frac{E_2}{\sinh^2 \phi}, & F^{(0,2)} &= \frac{1}{32} \frac{E_2}{\sinh^2 \phi}, \\ F^{(3,0)} &= \frac{1}{11520} \frac{5E_2^2 + 7E_4}{\sinh^2 \phi} + \frac{1}{11520} \frac{5E_2^2 + 13E_4}{\sinh^4 \phi}, \\ F^{(2,1)} &= \frac{1}{11520} \frac{35E_2^2 + 37E_4}{\sinh^2 \phi} + \frac{1}{2880} \frac{10E_2^2 + 17E_4}{\sinh^4 \phi}, \\ F^{(1,2)} &= \frac{1}{3840} \frac{25E_2^2 + 17E_4}{\sinh^2 \phi} + \frac{1}{11520} \frac{95E_2^2 + 94E_4}{\sinh^4 \phi}, \\ F^{(0,3)} &= \frac{1}{768} \frac{3E_2^2 + E_4}{\sinh^2 \phi} + \frac{1}{384} \frac{2E_2^2 + E_4}{\sinh^4 \phi}. \end{aligned} \quad (3.14)$$

We checked that these results are consistent with the previous results [14] about refined topological string amplitudes for the local $\frac{1}{2}\text{K3}$ at low genus.

Let us next study how much the modular anomaly equation (2.17) constrains the forms of $F^{(n,g)}$. First, $F^{(n,g)}$ with $n + g = 2$, presented in (3.13), are uniquely determined by the

modular anomaly equation, given the explicit forms of $F^{(n,g)}$ with $n+g < 2$. $F^{(n,g)}$ with $n+g \geq 3$ are not determined solely by the modular anomaly equation. As we mentioned, $F^{(n,g,h)}(\tau)$ is a quasi modular form of weight $2n+2g-2$. Its anomalous part is completely determined by the modular anomaly equation, but its modular part contains $\lceil (n+g)/6 \rceil$ (or $\lceil (n+g)/6 \rceil - 1$ if $n+g \equiv 2 \pmod{6}$) undetermined coefficients.

There are several ways to fix these coefficients. For example, in the next section we will study q -expansion of the BPS index. Full information on the leading order part F^{pert} completely fixes the forms of $F^{(n,g,h)}$ with $n+g \leq 6$, each of which contains at most one undetermined coefficient. We expect that the data of higher order coefficients $F_k^{q\text{-inst}}$ with $k \leq N$, together with F^{pert} , will fix $F^{(n,g,h)}$ with $n+g \leq 6(N+1)$. We check it for $N=2$ in the unrefined case, namely, we are able to determine $F^{(0,g,h)}$ with $g \leq 18$. Using other data one can fix different slices of the BPS index. We take the results of the following sections in advance and summarize how far we are able to determine $F^{(n,g,h)}$ at present:

- $F^{(n,g,h)}$ with $n+g \leq 6$ and any h are determined using the explicit forms of F^{pert} presented in section 4.
- $F^{(0,g,h)}$ with $g \leq 18$ and any h are determined using the explicit forms of F^{pert} , $F_1^{q\text{-inst}}$, $F_2^{q\text{-inst}}$ in the unrefined case presented in section 4.
- $F^{(n,g,1)}$ with any n, g can be computed by expanding the elliptic genus of two E-strings Z_2 presented in section 5.
- $F^{(0,g,2)}$ with any g can be computed from the elliptic genera Z_2, Z_4 in the unrefined case presented in section 5.
- $F^{(n,g,n+g-1)}$ with any n, g can be determined by the Nekrasov partition function for the $N_f = 4$ theory as we will see in section 6.

We do not present a vast amount of results here. The computations are straightforward. In doing this, formulas (A.16)–(A.23) are useful.

4 Instanton expansion

In this section we study the expansion (2.14). This expansion is closely related to the instanton expansion of conformal gauge theories. We first analyze the leading order part in subsection 4.1 and then study instanton corrections in subsection 4.2.

4.1 Perturbative part

F^{pert} is the leading order coefficient of the q -expansion (2.14). We call it the perturbative part by analogy with that of the Nekrasov partition function for conformal gauge theories. It can also be viewed as the following limit of the free energy

$$F^{\text{pert}} = \lim_{q \rightarrow 0} F. \quad (4.1)$$

Since τ is proportional to the inverse of the radius of S^1 around which E-strings are wound, the above limit corresponds to the five-dimensional limit of the E-string theory on $\mathbb{R}^5 \times S^1$ under the $D_4 \oplus D_4$ twist.

Let us first describe our result. We make a conjecture that F^{pert} takes the following form

$$F^{\text{pert}} = - \sum_{n=1}^{\infty} \frac{\sin^2(n\epsilon_1/2) + \sin^2(n\epsilon_2/2) + \sin^2(n(\epsilon_1 + \epsilon_2)/2)}{n \sin(n\epsilon_1) \sin(n\epsilon_2)} e^{-2n\phi}. \quad (4.2)$$

Let us outline how we arrive at this form. First, the leading order part (i.e. the summand with $n = 1$) of (4.2) immediately follows from the explicit form of Z_2 , which we will see in the next section. Next, by expanding the explicit forms (3.10) of the unrefined amplitudes $F^{(0,g)}$ in $p = e^{-\phi}$, it is not difficult to notice that F^{pert} in the unrefined case must take the form

$$F^{\text{pert}}(\phi, \hbar, -\hbar) = \sum_{n=1}^{\infty} \frac{1}{2n \cos^2(n\hbar/2)} e^{-2n\phi}. \quad (4.3)$$

Given the above two results, (4.2) arises as the most natural form of F^{pert} .

We checked this conjecture against the genus expansion of full Z_4^{gen} in the unrefined case computed by the conventional method. More importantly, the above form reproduces the perturbative part of the Nekrasov partition function for the $SU(2)$ $N_f = 4$ theory, as we will show in section 6. The validity of the conjecture is therefore checked in two extreme limits: $\phi \rightarrow \pm\infty$ and $\phi \rightarrow 0$. This gives strong evidence to the conjecture. Nevertheless, it would still be very interesting if one could give a proof or a derivation for it.

It is useful to note that the above F^{pert} can be expressed as

$$F^{\text{pert}} = -\gamma_{\epsilon_1, \epsilon_2}(2\phi) - \gamma_{\epsilon_1, \epsilon_2}(2\phi - i\epsilon_1 - i\epsilon_2) + 8\gamma_{2\epsilon_1, 2\epsilon_2}(2\phi - i\epsilon_1 - i\epsilon_2), \quad (4.4)$$

where

$$\gamma_{\epsilon_1, \epsilon_2}(x) := \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-nx}}{(1 - e^{in\epsilon_1})(1 - e^{in\epsilon_2})}. \quad (4.5)$$

It is easy to see that $\gamma_{\epsilon_1, \epsilon_2}(x)$ can be rewritten as

$$\gamma_{\epsilon_1, \epsilon_2}(x) = \ln \prod_{k,l=1}^{\infty} \left(1 - q_1^{k-1} t_2^l e^{-x}\right) \quad (4.6)$$

with

$$q_1 := e^{i\epsilon_1}, \quad t_2 := e^{-i\epsilon_2}, \quad (4.7)$$

when $|q_1| < 1$, $|t_2| < 1$. By using this expression, the perturbative part can also be written as

$$Z^{\text{pert}} = \exp(F^{\text{pert}}) = \frac{\left[\prod_{k,l=1}^{\infty} \left(1 - q_1^{2k-1} t_2^{2l-1} p^2\right) \right]^8}{\left[\prod_{k,l=1}^{\infty} \left(1 - q_1^{k-1} t_2^l p^2\right) \right] \left[\prod_{k,l=1}^{\infty} \left(1 - q_1^k t_2^{l-1} p^2\right) \right]}. \quad (4.8)$$

One can clearly see that the perturbative part of the BPS index is made up of the topological string partition functions for the resolved conifold [20]. The above expression is very reminiscent of the perturbative part of the five-dimensional extension of the Nekrasov partition function for $SU(2)$ theory with $N_f = 4$ massless flavors. In fact, the denominator of the above Z^{pert} coincides with that of the perturbative part of the 5d Nekrasov partition function. However, the numerator of the latter function is $[\prod_{k,l=1}^{\infty} (1 - q_1^{k-1/2} t_2^{l-1/2} p)]^8$, which differs from the one we see in (4.8).¹ Interestingly, despite this difference, both trigonometric generalizations reduce to the same four-dimensional perturbative part, as we will see in section 6.

4.2 Instanton part

Let us next consider instanton corrections. For the sake of simplicity in the rest of this section we restrict ourselves to the unrefined case, namely, we focus on

$$F_n^{q\text{-inst}} = F_n^{q\text{-inst}}(\phi, \hbar, -\hbar). \quad (4.9)$$

As we mentioned in subsection 3.2, F^{pert} completely determines the forms of $F^{(0,g)}$ with $g \leq 6$. By expanding them in q , one obtains information about instanton corrections. In fact, the above data suffice to determine the form of the first instanton correction $F_1^{q\text{-inst}}$. This further gives us the data of $F^{(0,g)}$ with $g \leq 12$. Using these data (with several consistency conditions), we are also able to determine $F_2^{q\text{-inst}}$. Here we present the results:

$$F_1^{q\text{-inst}} = \frac{t(4t-3)y + t^3}{(1-t)(y+t)^2}, \quad (4.10)$$

$$\begin{aligned} F_2^{q\text{-inst}} = & [2t(6t-1)(2t-1)^2(4t-3)^2y^5 \\ & + t^2(2048t^7 - 10240t^6 + 23552t^5 - 29824t^4 \\ & + 21600t^3 - 8776t^2 + 1821t - 144)y^4 \\ & + t^3(8192t^7 - 40960t^6 + 90112t^5 - 109696t^4 \\ & + 78288t^3 - 32336t^2 + 7074t - 621)y^3 \\ & + t^4(28672t^7 - 135168t^6 + 272384t^5 - 302080t^4 \\ & + 197804t^3 - 75912t^2 + 15651t - 1314)y^2 \\ & + t^5(1024t^5 - 2944t^4 + 3288t^3 - 1736t^2 + 408t - 27)(4t-3)^2y \\ & + 2t^7(6t^2 - 8t + 3)(4t-3)^4] \\ & / [2(1-t)(2t-1)^2(y+t)^4(y+t(4t-3)^2)^2], \end{aligned} \quad (4.11)$$

where

$$y := \sinh^2 \phi, \quad t := \sin^2 \frac{\hbar}{2}. \quad (4.12)$$

¹Though the difference is small in appearance, it implies a qualitative difference in the low energy effective gauge theory. F^{pert} for the E-string theory under the $D_4 \oplus D_4$ twist yields a vanishing prepotential at genus zero, meaning that the effective gauge coupling does not vary. On the other hand, the perturbative part of the Nekrasov partition function for 5d $SU(2)$ $N_f = 4$ theory gives a nontrivial genus zero part, meaning that the effective coupling depends on the expectation value ϕ .

As we mentioned repeatedly, the BPS index Z is regarded as the trigonometric generalization of the Nekrasov partition function for the 4d $SU(2)$ $N_f = 4$ theory. In other words, Z reduces to the Nekrasov partition function by taking an appropriate limit. In this limit, $F_k^{q\text{-inst}}$ reduce to their counterparts, which will be denoted by $F_{YM,k}^{q\text{-inst}}$. By analyzing the Nekrasov partition function, it can be easily verified that $F_{YM,k}^{q\text{-inst}}$ determines the contribution of up to and including $2k + 1$ instantons. We thus expect that our results (4.2), (4.10) and (4.11) give the full information of up to and including five ‘instanton’ contributions in the unrefined case.

Note that the denominators of the above $F_k^{q\text{-inst}}$ can be expressed as

$$\left[F_1^{q\text{-inst}}\right]_{\text{denom}} = \cos^2 \frac{\hbar}{2} \left(\sinh^2 \phi + \sin^2 \frac{\hbar}{2} \right)^2, \quad (4.13)$$

$$\left[F_2^{q\text{-inst}}\right]_{\text{denom}} = 2 \cos^2 \frac{\hbar}{2} \cos^2 \frac{2\hbar}{2} \left(\sinh^2 \phi + \sin^2 \frac{\hbar}{2} \right)^4 \left(\sinh^2 \phi + \sin^2 \frac{3\hbar}{2} \right)^2. \quad (4.14)$$

This gives us a hint about how to generalize the 4d Nekrasov partition function to construct a closed Nekrasov-type expression for the BPS index Z .

5 Winding number expansion

In this section we consider the expansion (2.15) and study elliptic genera of multiple E-strings. The modular anomaly equation (2.18) works as a powerful tool in determining the form of the elliptic genera. It is expected that elliptic genera of multiple E-strings are written in terms of Jacobi theta functions and elliptic functions in ϵ_1, ϵ_2 [16]. Recall that E_2 never appears in the expansions of Weierstrass elliptic functions. The only source of the modular anomaly is the E_2 appearing in the theta function. See (A.20)–(A.23) and (A.16) for further details.

Let us first consider the leading coefficient Z_2 . The general elliptic genus of two E-strings Z_2^{gen} was calculated recently [16]. Thus the explicit form of Z_2 is derived by simply substituting (2.11) into it. We find that the result can be expressed in a very simple form as follows:

$$Z_2 = \frac{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_1 + \epsilon_2)}{\eta^3\vartheta_1(2\epsilon_1)\vartheta_1(2\epsilon_2)} \frac{\wp'(\epsilon_1) - \wp'(\epsilon_2)}{\wp(\epsilon_1) - \wp(\epsilon_2)} \quad (5.1)$$

$$= 2 \frac{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_3)}{\eta^3\vartheta_1(2\epsilon_1)\vartheta_1(2\epsilon_2)} (\zeta(\epsilon_1) + \zeta(\epsilon_2) + \zeta(\epsilon_3)), \quad (5.2)$$

where

$$\epsilon_3 := -\epsilon_1 - \epsilon_2. \quad (5.3)$$

Here $\zeta(z), \wp(z)$ are Weierstrass elliptic functions (see appendix A). The above explicit form completely determines $F^{(n,g,1)}$ (defined for $n + g \geq 2$) through

$$Z_2 = 4 \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1\epsilon_2)^{g-1} F^{(n,g,1)}. \quad (5.4)$$

In the unrefined case, the above expression further reduces to a remarkably simple form:

$$Z_2(\tau, \hbar, -\hbar) = 2 \frac{\vartheta_1(\hbar)^2}{\vartheta_1(2\hbar)^2}. \quad (5.5)$$

Let us next consider elliptic genera Z_{2n} with $n \geq 2$. For the sake of simplicity in the rest of this subsection we only consider the unrefined case

$$Z_{2n} = Z_{2n}(\tau, \hbar, -\hbar). \quad (5.6)$$

We make the following ansatz for the general form of Z_{2n} in the unrefined case:

$$Z_{2n} = \frac{1}{\eta^{8(n^3-n)}} \frac{\vartheta_1(\hbar)^{(8n^3-2n)/3}}{\prod_{k=1}^n \vartheta_1(2k\hbar)^2} \left[\text{polynomial in } \wp(\hbar), E_4, E_6 \text{ of weight } \frac{8(n^3-n)}{3} \right]. \quad (5.7)$$

The denominator of the ansatz is inferred naturally from the pole structure of F^{pert} given in the last section. The rest of the ansatz is determined by the modular anomaly equation (2.18) and other expected modular properties.

For $n = 2$, the above ansatz contains 10 unknown coefficients. By using the data of F^{pert} and $F_1^{q\text{-inst}}$ given in the last section, these coefficients are completely fixed. We thus obtain

$$Z_4 = \frac{1}{2\eta^{48}} \frac{\vartheta_1(\hbar)^{20}}{\vartheta_1(4\hbar)^2 \vartheta_1(2\hbar)^2} (72\wp'^4\wp^2 - 18\wp''^2\wp'^2\wp + 2\wp''\wp'^4 + \wp''^4), \quad (5.8)$$

where $\wp = \wp(\hbar)$, $\wp' = \wp'(\hbar)$, $\wp'' = \wp''(\hbar)$. It turns out that the polynomial part is written in a concise form by using \wp, \wp'', \wp'^2 as the generators instead of \wp, E_4, E_6 . The above result of Z_4 passed several nontrivial consistency checks.

For $n = 3$, the above ansatz for Z_6 contains 102 unknown coefficients. The explicit forms of F^{pert} and $F_k^{q\text{-inst}}$ with $k = 1, 2$, which are the best available data at present, are not enough to determine Z_6 completely: these data give 78 relations among the coefficients, but 24 parameters are left to be determined by other means.

6 Relation to 4d $\mathcal{N} = 2$ SU(2) $N_f = 4$ theory

The reduced BPS index Z can be viewed as a trigonometric generalization of the Nekrasov partition function for 4d $\mathcal{N} = 2$ SU(2) gauge theory with $N_f = 4$ massless flavors. This means that Z reduces to the Nekrasov partition function in a certain limit. In this section we elucidate how this occurs. It was shown in [21] that the Seiberg-Witten curve for the E-string theory reproduces that for the $N_f = 4$ theory by taking an appropriate limit. Here we generalize this correspondence to the level of the refined BPS index.

6.1 Nekrasov partition function

The Nekrasov partition function for 4d $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with $N_f = 4$ massless flavors is given (up to a ‘U(1) factor’ which we will mention later) by [10, 11]

$$Z_{\text{YM}} = Z_{\text{YM}}^{\text{pert}} Z_{\text{YM}}^{\text{inst}}. \quad (6.1)$$

The perturbative part is given as $Z_{\text{YM}}^{\text{pert}} = \exp F_{\text{YM}}^{\text{pert}}$ with

$$F_{\text{YM}}^{\text{pert}} = -\gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}(2a) - \gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}(2a - \epsilon_1 - \epsilon_2) + 8\gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}\left(a - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2}\right), \quad (6.2)$$

where

$$\gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}(x) := \frac{d}{ds} \bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)}. \quad (6.3)$$

The instanton part is given by

$$Z_{\text{YM}}^{\text{inst}} = \sum_{\mathbf{R}} q_0^{|\mathbf{R}|} \prod_{k,l=1}^2 \prod_{(i,j) \in R_k} \prod_{\alpha=1}^2 \frac{a_k + (j - \frac{1}{2})\epsilon_1 + (i - \frac{1}{2})\epsilon_2}{a_k - a_l + (\mu_{k,i} - j + \delta_{1,\alpha})\epsilon_1 - (\mu_{l,j}^\vee - i + \delta_{2,\alpha})\epsilon_2}, \quad (6.4)$$

where

$$a_1 = a, \quad a_2 = -a. \quad (6.5)$$

Here $\mathbf{R} = (R_1, R_2)$ denotes a pair of partitions and $|\mathbf{R}|$ the total number of boxes in the Young diagrams of R_1, R_2 . $\mu_{k,i}$ ($\mu_{k,j}^\vee$) denotes the length of i th row (j th column) of the Young diagram of R_k . The sum is taken over all possible partitions \mathbf{R} (including the empty partition). The set of indices (i, j) runs over the coordinates of all boxes in the Young diagram of R_k .

The first two terms of $F_{\text{YM}}^{\text{pert}}$ represent the contribution from the vector multiplet, where the argument in the second term should be shifted as above (see e.g. [22]). The last term of $F_{\text{YM}}^{\text{pert}}$ represents the contribution from the four massless fundamental matters. Note that we have taken account of the shift of the mass parameters by $(\epsilon_1 + \epsilon_2)/2$ which is needed in order for the partition function to have good modular properties [23]. It is now well known that the UV gauge coupling τ_{UV} appearing in the Nekrasov partition function through $q_0 = \exp(2\pi i \tau_{\text{UV}})$ is not identical with the IR gauge coupling τ represented by the complex modulus of the Seiberg-Witten curve [24]. q_0 and $q = \exp(2\pi i \tau)$ are related to each other by [12]

$$\begin{aligned} q_0 &= \frac{\vartheta_2^4}{\vartheta_3^4} \\ &= 16q^{1/2} - 128q + 704q^{3/2} - 3072q^2 + \mathcal{O}(q^{5/2}), \end{aligned} \quad (6.6)$$

or

$$q = \frac{1}{256}q_0^2 + \frac{1}{256}q_0^3 + \frac{29}{8192}q_0^4 + \frac{13}{4096}q_0^5 + \mathcal{O}(q_0^6). \quad (6.7)$$

The free energy of the above partition function can be expanded as

$$F_{\text{YM}} \equiv \ln Z_{\text{YM}} = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1 \epsilon_2)^{g-1} (-a^2)^{-n-g+1} F_{\text{YM}}^{(n,g)}. \quad (6.8)$$

One can check that

$$F_{\text{YM}}^{(0,0)} = \ln \frac{q_0}{q^{1/2}}, \quad F_{\text{YM}}^{(1,0)} = -\frac{1}{2} \ln 2a + \ln \frac{\vartheta_3^2}{\vartheta_4} + \frac{\ln 2}{3}, \quad F_{\text{YM}}^{(0,1)} = -\frac{1}{2} \ln 2a + \frac{2 \ln 2}{3}. \quad (6.9)$$

The second term $\ln(\vartheta_3^2/\vartheta_4)$ in the expression of $F_{\text{YM}}^{(1,0)}$ may be regarded as the ‘U(1) factor,’ which should be stripped off when the difference between SU(2) and U(2) gauge groups matters [22].² However, such a difference within $F_{\text{YM}}^{(1,0)}$ is not relevant to our discussion below. It is known that $F_{\text{YM}}^{(n,g)}$ ($n+g \geq 2$) is a quasi modular form in τ of weight $2n+2g-2$ [12]. Explicit forms of them with small $n+g$ are found as [23]

$$\begin{aligned} F_{\text{YM}}^{(2,0)} &= \frac{E_2}{96}, & F_{\text{YM}}^{(1,1)} &= \frac{E_2}{24}, & F_{\text{YM}}^{(0,2)} &= \frac{E_2}{32}, \\ F_{\text{YM}}^{(3,0)} &= \frac{5E_2^2 + 13E_4}{11520}, & F_{\text{YM}}^{(2,1)} &= \frac{10E_2^2 + 17E_4}{2880}, \\ F_{\text{YM}}^{(1,2)} &= \frac{95E_2^2 + 94E_4}{11520}, & F_{\text{YM}}^{(0,3)} &= \frac{2E_2^2 + E_4}{384}. \end{aligned} \quad (6.10)$$

6.2 Limit

We claim that the above Nekrasov partition function Z_{YM} is reproduced from the reduced BPS index Z by rescaling the variables as

$$\phi \rightarrow \beta a, \quad \epsilon_\alpha \rightarrow -i\beta\epsilon_\alpha \quad (6.11)$$

and taking the limit $\beta \rightarrow 0$. Recall that the free energy $F = \ln Z$ admits the genus expansion (2.13) with coefficients in the form (3.11). By taking the above limit, the higher genus part of the free energy becomes

$$\begin{aligned} F|_{n+g \geq 2} &= \sum_{n+g \geq 2} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1 \epsilon_2)^{g-1} (-i\beta)^{2n+2g-2} \sum_{h=1}^{n+g-1} \frac{F^{(n,g,h)}(\tau)}{\sinh^{2h} \beta a} \\ &\xrightarrow{\beta \rightarrow 0} \sum_{n+g \geq 2} (\epsilon_1 + \epsilon_2)^{2n} (-\epsilon_1 \epsilon_2)^{g-1} (-a^2)^{-n-g+1} F^{(n,g,n+g-1)}. \end{aligned} \quad (6.12)$$

This takes the same form as (6.8). We find that

$$F^{(n,g,n+g-1)}(\tau) = F_{\text{YM}}^{(n,g)}(\tau) \quad (6.13)$$

for $n+g \geq 2$. We verified this equality for $2 \leq n+g \leq 6$ by explicit calculation. We do not have a complete proof of this identity. In the rest of this section we present some evidence supporting further this conjectured identity.

Let us first show that the modular anomaly equation (2.17) reduces to that for the $N_f = 4$ theory. We substitute the general form of $F^{(n,g)}$ (3.11) and $\phi = \beta a$ into (2.17). We then multiply $\beta^{2n+2g-2}$ to both sides and take the limit $\beta \rightarrow 0$. The equation becomes

$$\begin{aligned} \frac{\partial}{\partial E_2} \frac{F^{(n,g,n+g-1)}}{a^{2n+2g-2}} &= \frac{1}{24} \sum_{n_1=0}^n \sum_{g_1=0}^g \left(\frac{\partial}{\partial a} \frac{F^{(n_1,g_1,n_1+g_1-1)}}{a^{2n_1+2g_1-2}} \right) \left(\frac{\partial}{\partial a} \frac{F^{(n-n_1,g-g_1,n-n_1+g-g_1-1)}}{a^{2n-2n_1+2g-2g_1-2}} \right) \\ &\quad + \frac{1}{24} \frac{\partial^2}{\partial a^2} \frac{F^{(n,g-1,n+g-2)}}{a^{2n+2g-4}}. \end{aligned} \quad (6.14)$$

²To be precise, the U(1) factor proposed in [22] increases $F_{\text{YM}}^{(1,0)}$ by $\ln(\vartheta_3^2/\vartheta_4)$. This is very similar, but not identical to the term $\ln(\vartheta_3^2/\vartheta_4)$ in (6.9). (We verified (6.9) up to order q_0^{15} .)

Here $F^{(n,g,n+g-1)}$ for $n+g \leq 1$ are understood as

$$F^{(0,0,-1)} = 0, \quad F^{(1,0,0)} = F^{(0,1,0)} = -\frac{1}{2} \ln 2a - \frac{1}{2} \ln \beta. \quad (6.15)$$

Following [23], we introduce the rescaled amplitudes $f^{(n,g)}$ by

$$\begin{aligned} f^{(0,0)} &= 0, & f^{(1,0)} &= f^{(0,1)} = \frac{1}{4}, \\ f^{(n,g)} &= (n+g-1)F^{(n,g,n+g-1)} & \text{for } n+g &\geq 2 \end{aligned} \quad (6.16)$$

and $f^{(n,g)} = 0$ if $n < 0$ or $g < 0$. The above equation is then written in terms of $f^{(n,g)}$ as

$$\partial_{E_2} f^{(n,g)} = \frac{n+g-1}{6} \left[\sum_{n_1=0}^n \sum_{g_1=0}^g f^{(n_1,g_1)} f^{(n-n_1,g-g_1)} + \left(n+g-\frac{3}{2}\right) f^{(n,g-1)} \right]. \quad (6.17)$$

This is precisely the modular anomaly equation for the $N_f = 4$ theory [23].

Let us next show that the perturbative part of the BPS index of E-strings reproduces that of the $N_f = 4$ theory in the limit $\beta \rightarrow 0$. By starting from the expression (4.4) with (4.5), it is not difficult to show that

$$\begin{aligned} \lim_{\beta \rightarrow 0} F^{\text{pert}}(\beta a, -i\beta\epsilon_1, -i\beta\epsilon_2) \\ = -\gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}(2a) - \gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}(2a - \epsilon_1 - \epsilon_2) + 8\gamma_{2\epsilon_1, 2\epsilon_2}^{\text{YM}}(2a - \epsilon_1 - \epsilon_2) + \frac{\epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2}{2\epsilon_1\epsilon_2} \ln \beta. \end{aligned} \quad (6.18)$$

Here $\ln \beta$ is understood as an infinite constant. By performing the series expansion in ϵ_1, ϵ_2 , it is easy to check that

$$\gamma_{2\epsilon_1, 2\epsilon_2}^{\text{YM}}(2a - \epsilon_1 - \epsilon_2) = \gamma_{\epsilon_1, \epsilon_2}^{\text{YM}}\left(a - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2}\right) - \frac{\ln 2}{2} \frac{a^2}{\epsilon_1\epsilon_2} + \frac{\ln 2}{24} \frac{\epsilon_1^2 + \epsilon_2^2}{\epsilon_1\epsilon_2}. \quad (6.19)$$

Therefore, F^{pert} in the $\beta \rightarrow 0$ limit reproduces $F_{\text{YM}}^{\text{pert}}$

$$\lim_{\beta \rightarrow 0} F^{\text{pert}}(\beta a, -i\beta\epsilon_1, -i\beta\epsilon_2) = F_{\text{YM}}^{\text{pert}} + \text{shifts}. \quad (6.20)$$

Here *shifts* denote constant shifts of $F_{\text{YM}}^{(n,g)}$ with $n+g \leq 1$. Hence, the conjectured identity (6.13) in the limit $q \rightarrow 0$ has been proved.

Finally, let us verify that the instanton expansion performed in section 4 is also consistent with the Nekrasov partition function. One can define the counterpart of $F_k^{q\text{-inst}}$ studied in section 4. As we define Z as the BPS index rather than the topological string partition function, Z does not have the ‘classical part’, as opposed to Z_{YM} . In order to make the comparison accurate, we have to subtract such part at low genus from F_{YM} . More specifically, we define $F_{\text{YM},k}^{q\text{-inst}}$ by

$$\begin{aligned} F_{\text{YM}}^{\text{inst}}|_{n+g \geq 2} &= \ln Z_{\text{YM}}^{\text{inst}} - \frac{a^2}{\epsilon_1\epsilon_2} \ln \frac{q_0}{2^4 q^{1/2}} + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} \ln \frac{\vartheta_3^2}{\vartheta_4} \\ &=: \sum_{k=1}^{\infty} q^k F_{\text{YM},k}^{q\text{-inst}}(a, \epsilon_1, \epsilon_2). \end{aligned} \quad (6.21)$$

By evaluating the Nekrasov partition function (6.4) up to and including the terms of order q_0^5 , we obtain

$$F_{\text{YM},1}^{q\text{-inst}}(a, \hbar, -\hbar) = \frac{12\hbar^2 a^2}{(2a - \hbar)^2 (2a + \hbar)^2}, \quad (6.22)$$

$$F_{\text{YM},2}^{q\text{-inst}}(a, \hbar, -\hbar) = \frac{18\hbar^2 a^2 (512a^8 - 1024\hbar^2 a^6 + 1104\hbar^4 a^4 - 584\hbar^6 a^2 + 27\hbar^8)}{(2a - \hbar)^4 (2a + \hbar)^4 (2a - 3\hbar)^2 (2a + 3\hbar)^2}. \quad (6.23)$$

One can easily check that $F_k^{q\text{-inst}}$ presented in (4.10), (4.11) indeed reduce to the above expressions

$$\lim_{\beta \rightarrow 0} F_k^{q\text{-inst}}(\beta a, -i\beta\hbar, i\beta\hbar) = F_{\text{YM},k}^{q\text{-inst}}(a, \hbar, -\hbar). \quad (6.24)$$

7 Discussion

In this paper we studied a reduced BPS index of E-strings. This is obtained by evaluating the refined BPS index of E-strings at special values of E_8 Wilson line parameters that correspond to breaking the global E_8 symmetry down to $D_4 \oplus D_4$. The index admits three kinds of expansions, each of which corresponds to an entirely different physical picture. We clarified the structure of expansions and determined some expansion coefficients. We elucidated in detail how the reduced BPS index reduces to the Nekrasov partition function for 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with $N_f = 4$ massless flavors.

It would be interesting to generalize the present study to the case of the $SU(2)$ $N_f = 4$ theory with massive flavors. At the level of the Seiberg-Witten curve, it is already known [21] how to identify the E_8 Wilson line parameters with the masses of flavors in the $SU(2)$ $N_f = 4$ theory. Interestingly, the identification has to take a quite nontrivial form in order to keep the modular properties intact.

A very simple form of Nekrasov-type expression is available for the Seiberg-Witten prepotential for the E-string theory [21, 25, 26]. The expression however does not reproduce the higher genus part. An appropriate modification or some alternative formula is anticipated. The present study may open up a way of constructing the all genus Nekrasov-type partition function for the E-string theory.

The reduced BPS index of E-strings gives a novel trigonometric generalization of the Nekrasov partition function for the $SU(2)$ $N_f = 4$ theory. As is well known, this Nekrasov partition function is identified with a certain Virasoro conformal block through Alday-Gaiotto-Tachikawa relation [22]. It would be extremely interesting if there exists an interpretation of the BPS index of E-strings on the CFT side.

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A Conventions of special functions and useful formulas

The Jacobi theta functions are defined as

$$\vartheta_1(z, \tau) := i \sum_{n \in \mathbb{Z}} (-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \quad (\text{A.1})$$

$$\vartheta_2(z, \tau) := \sum_{n \in \mathbb{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \quad (\text{A.2})$$

$$\vartheta_3(z, \tau) := \sum_{n \in \mathbb{Z}} y^n q^{n^2/2}, \quad (\text{A.3})$$

$$\vartheta_4(z, \tau) := \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{n^2/2}, \quad (\text{A.4})$$

where

$$y = e^{iz}, \quad q = e^{2\pi i \tau}. \quad (\text{A.5})$$

In this paper we adopt slightly different convention for the theta functions as compared to our previous works [17, 21, 25, 26]. The above theta functions are related to those in our previous convention by

$$\vartheta_k(z, \tau) = \vartheta_k^{\text{previous}}\left(\frac{z}{2\pi}, \tau\right). \quad (\text{A.6})$$

We often use the following abbreviated notation

$$\vartheta_k(z) := \vartheta_k(z, \tau), \quad \vartheta_k := \vartheta_k(0, \tau). \quad (\text{A.7})$$

The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{A.8})$$

The Eisenstein series are given by

$$E_{2n}(\tau) = 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{2n-1} q^k}{1 - q^k} \quad (\text{A.9})$$

for $n \in \mathbb{Z}_{>0}$. The Bernoulli numbers B_k are defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k. \quad (\text{A.10})$$

We often abbreviate $\eta(\tau)$, $E_{2n}(\tau)$ as η , E_{2n} respectively.

The Weierstrass elliptic functions are defined as

$$\sigma(z; 2\omega_1, 2\omega_3) := z \prod_{(m,n) \in \mathbb{Z}_{\neq(0,0)}^2} \left(1 - \frac{z}{\Omega_{m,n}}\right) \exp \left[\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2} \right], \quad (\text{A.11})$$

$$\zeta(z; 2\omega_1, 2\omega_3) := \frac{1}{z} + \sum_{(m,n) \in \mathbb{Z}_{\neq(0,0)}^2} \left[\frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right], \quad (\text{A.12})$$

$$\wp(z; 2\omega_1, 2\omega_3) := \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}_{\neq(0,0)}^2} \left[\frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \quad (\text{A.13})$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$. In this paper we always set $2\omega_1 = 2\pi$, $2\omega_3 = 2\pi\tau$ and use the following abbreviated notation

$$\sigma(z) := \sigma(z; 2\pi, 2\pi\tau), \quad \zeta(z) := \zeta(z; 2\pi, 2\pi\tau), \quad \wp(z) := \wp(z; 2\pi, 2\pi\tau). \quad (\text{A.14})$$

Note that

$$\frac{d}{dz} \ln \sigma(z) = \zeta(z), \quad \frac{d}{dz} \zeta(z) = -\wp(z). \quad (\text{A.15})$$

The sigma function $\sigma(z)$ is related to the Jacobi theta function $\vartheta_1(z)$ by

$$\vartheta_1(z) = e^{-\frac{1}{24}E_2 z^2} \eta^3 \sigma(z). \quad (\text{A.16})$$

To expand the above elliptic functions in $q = e^{2\pi i\tau}$, the following formulas are useful:

$$\vartheta_1(z) = 2q^{1/12} \eta(\tau) \sin \frac{z}{2} \prod_{n=1}^{\infty} (1 - 2q^n \cos z + q^{2n}), \quad (\text{A.17})$$

$$\zeta(z) = \frac{z}{12} + \frac{1}{2} \cot \frac{z}{2} + 2 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} (\sin nz - nz), \quad (\text{A.18})$$

$$\wp(z) = -\frac{1}{12} + \frac{1}{4 \sin^2(z/2)} + 4 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \sin^2 \frac{nz}{2}. \quad (\text{A.19})$$

The expansions of the elliptic functions in z about $z = 0$ are obtained as

$$\sigma(z) = z \exp \left(- \sum_{n=1}^{\infty} \frac{c_n}{(2n+1)(2n+2)} z^{2n+2} \right), \quad (\text{A.20})$$

$$\zeta(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c_n}{2n+1} z^{2n+1}, \quad (\text{A.21})$$

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n z^{2n}, \quad (\text{A.22})$$

where c_n are determined by the recurrence relation

$$\begin{aligned} c_1 &= \frac{E_4}{240}, & c_2 &= \frac{E_6}{6048}, \\ c_n &= \frac{3}{(n-2)(2n+3)} \sum_{k=1}^{n-2} c_k c_{n-k-1} & (n \geq 3). \end{aligned} \quad (\text{A.23})$$

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